THE (NON-EQUIVARIANT) HOMOLOGY OF THE LITTLE DISKS OPERAD

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1. Introduction

This expository paper aims to be a gentle introduction to the topology of configuration spaces, or equivalently spaces of little disks. The pantheon of topological spaces which beginning graduate students see is limited – spheres, projective spaces, products of such, perhaps some spaces such as Lie groups, Grassmannians or knot complements. We would like for Euclidean configuration spaces to be added to this list. We aim for this article to be appropriate for someone who knows only basic homology and cohomology theory. The one exception to this rule will be the light use of a spectral sequence argument, for an upper bound.

Any space from the pantheon has rich associated combinatorial and algebraic structure. For example, the relationship between cohomology of projective spaces and Grassmannians is encoded by the structure of symmetric polynomials. In the case of configuration spaces, we are led to study graphs, trees, Jacobi and Arnold identities, and ultimately the Poisson operad. One goal of this paper is to explain the topology which leads to the configuration pairing between graphs and trees, developed purely combinatorially in [21]. That this pairing arises as that between canonical spanning sets for homology and cohomology of configuration spaces is a new result. Another goal is to prepare a reader for further study of the theory of operads by giving a thorough understanding of the disks operad from topology, the Poisson operad from algebra, and the fact that they are related through homology. We also bring in recently developed ingredients such as canonical compactifications of configuration spaces and submanifolds defined by collinearities. These new results and points of view, and our elementary development, differentiate this paper from expositions such as [5, 6, 9, 4, 1].

The plan of the paper is as follows. First in Section 2 we associate a class in the homology of configuration spaces to any forest, as the fundamental class of a submanifold homeomorphic to a torus, and then develop relations between such classes. Then in Section 3 we associate a cohomology class which is pulled back from a map to a torus to any graph. Section 4 gives the main new results, identifying the evaluation of our graphical cohomology classes on the forest homology classes with a combinatorially defined pairing between graphs and trees. This pairing is useful in a number of contexts, for example in simultaneously understanding free Lie algebras and coalgebras [22], but is not widely known. In Section 5 we give an informal "examples-first" development of operads, complementary to others in this volume, in order to be self-contained. In Section 6, in particular Theorem 6.3, we prove the well-known result that the homology of the little disks operad is the graded Poisson operad. Instead of the usual practice of waiving our hands at the operad structure maps, we are able to provide a complete argument by arguing on cohomology instead. At the end of most sections we give some (incomplete) historical notes.

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2. Homology generators of configuration spaces

In this section we construct homology classes for configuration spaces, all represented by submanifolds homeomorphic to tori. The term "configuration space" is used in different ways by different subfields. We use the term as is standard in algebraic topology, as the space of distinct labeled points in some ambient space.

Definition 2.1. The configuration space of n distinct points in a space X, denoted $Conf_n(X)$, is the subspace of the product $X^{\times n}$ defined as follows

$$\{(x_1,\ldots,x_n)\in X^{\times n}|\,x_i\neq x_j\,\text{if }i\neq j\}.$$

We will sometimes abbreviate (x_1, \dots, x_n) as \mathbf{x} . We focus on the case in which X is a Euclidean space \mathbb{R}^d . This configuration space models all possible simultaneous positions of n distinct planets or particles. This space as a whole may be visualized through linear algebra, starting with the ambient Euclidean space \mathbb{R}^{nd} and removing the hyperplanes where some $x_i = x_j$. Indeed, the Euclidean configuration spaces are special important cases of complements of hyperplane arrangements.

Our strategy to compute the homology and cohomology of these spaces is to "just get our hands on things." When n = 2 we have the following.

Proposition 2.2. The configuration space $Conf_2(\mathbb{R}^d)$ is homotopy equivalent to S^{d-1} . Thus the homology of $Conf_2(\mathbb{R}^d)$ is free, rank one in dimensions 0 and d-1, and zero otherwise.

Proof. We include S^{d-1} into $\operatorname{Conf}_2(\mathbb{R}^d)$ as a deformation retract. With an eye towards generalization, define the subspace $P_{1,2}$ of $\operatorname{Conf}_2(\mathbb{R}^d)$ as $\{(x_1,x_2) \mid x_1=-x_2 \text{ and } |x_i|=1\}$. The deformation retract onto

this subspace sends (x_1, x_2) to $\left(\frac{x_1-m}{|x_1-m|}, \frac{x_2-m}{|x_2-m|}\right)$, where $m = \frac{x_1+x_2}{2}$. The homotopy between this retraction and the identity map is given by a straight-line homotopy.

We also deduce that the generating cycle in $H_{d-1}\left(\operatorname{Conf}_2(\mathbb{R}^d)\right)$ is the image of the fundamental class of the sphere by the map which sends $v \in S^{d-1}$ to (v, -v), parameterizing $P_{1,2}$. Dually, $H^{d-1}\left(\operatorname{Conf}_2(\mathbb{R}^d)\right)$ is pulled back from the sphere by the given retraction. More geometrically, a generator of cohomology is Lefshetz dual to the submanifold (x_1, x_2) such that $\frac{x_1 - x_2}{|x_1 - x_2|}$ is say the north pole S^{d-1} . This cohomology thus evaluates on some d-1 dimensional cycle by counting with signs the number of configurations parameterized by that cycle for which " x_1 lies over x_2 ."

For the general case the language of solar systems is suggestive, as Fred Cohen likes to point out. In the n=2 case, the fundamental cycle had the "planets" x_1 and x_2 "orbiting" their center of mass. For n>2 we can build further cycles by having that "system" orbit the common center of mass with some other

planet or system of planets. Each time we build a system, it is possible (if there are more planets around) to put that in an orbit with another system to create a more complicated one. Such systems, which are more difficult to formalize than to visualize (see Figure 1), are naturally indexed by trees.

Definition 2.3. (1) An S-tree is an isotopy class of acyclic graph whose vertices are either trivalent or univalent, with a distinguished univalent vertex called the root, embedded in the upper half-plane

with the root at the origin – for example $T = \underbrace{ \begin{smallmatrix} 1 & 73 \\ 2 & 6 \end{smallmatrix}}_{2 & 6}$. Univalent vertices other than the root are called leaves, and they are labeled by a subset S of some set $\mathbf{n} = \{1, \dots, n\}$. Trivalent vertices are also called internal vertices.

- (2) The height of a vertex in an S-tree, denoted h(v), is the number of edges between v and the root. Edges which connect a vertex to higher vertices are called outgoing.
- (3) To define a subtree of T, take some vertex v and all of the vertices and edges above it. Restrict the ambient embedding in the upper half plane, and add a root edge from v to the origin, to obtain a tree we call T_v . Moreover, let T_v^L be the subtree associated to the left vertex over v, and similarly T_v^R be the right subtree over v.
- (4) We say that v is above or over w if w lies in the shortest path from v to the root. Define a total order on the vertices of T so that v < w if v lies over the left outgoing edge of w and v > w if it lies over the right outgoing edge of w. This total ordering can be realized as a left to right ordering of an appropriate planar embedding.

We now define the "centers of mass" for our systems and sub-systems.

Definition 2.4. The center $c(\mathbf{x}, T)$ of a configuration \mathbf{x} with respect to a tree T is defined inductively by $c(\mathbf{x}, T_v) = \frac{1}{2} \left(c(\mathbf{x}, T_v^L) + c(\mathbf{x}, T_v^R) \right)$, if T has at least one internal vertex. If T consists of only a leaf labeled by i, then $c(\mathbf{x}, T) = x_i$.

Finally, we can define the systems as ones where planets in a (sub)system are of a prescribed distance from the center of mass. Fix (for the moment) an $\varepsilon < \frac{1}{3}$.

Definition 2.5. Given an S-tree T, the (planetary system) P_T is the submanifold of all $\mathbf{x} = (x_1, \dots, x_n)$ such that:

- (1) $c(\mathbf{x}, T) = 0$.
- (2) For any vertex v of T, $d\left(c(\mathbf{x}, T_v^L), c(\mathbf{x}, T_v)\right) = \varepsilon^{h(v)} = d\left(c(\mathbf{x}, T_v), c(\mathbf{x}, T_v^R)\right)$, where d is the standard Euclidean distance function.
- (3) If $i \notin S$, x_i is fixed as some point "at infinity."

We picture these submanifolds as in Figure 1, which illustrates the case of $T = \frac{2.6}{1.00}$

One configuration in this submanifold is illustrated by the \bullet , which are labeled by x_i . The rest of the family is indicated by drawing some of the circular orbits where points in these configurations occur. The centers of this configuration, namely the points $c(\mathbf{x}, T_v)$, are indicated by \circ . In any configuration in P_T , the points x_4 and x_5 occur where they are indicated.

We will use P_T to define a homology class but to do so integrally, rather than only with $\mathbb{Z}/2$ coefficients, it must be oriented. We orient P_T by parametrizing it through a map from a torus.

Definition 2.6. By abuse of notation, let $P_T: (S^{d-1})^{\times |T|} \to \operatorname{Conf}_n(\mathbb{R}^d)$, where |T| is the number of internal vertices of T, send $(u_{v_1}, \ldots, u_{v_{|T|}})$ to (x_1, \ldots, x_n) , where

$$x_i = \sum_{v_j \text{ below leaf } i} \pm \varepsilon^{h_j} u_{v_j}.$$

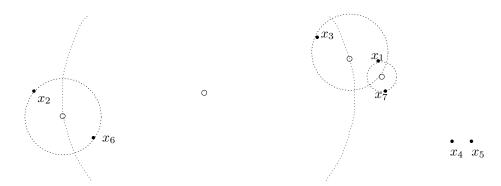


Figure 1. An illustration of P_T

Here the sum is taken over all vertices v_j which lie on the path from the leaf labeled by i to the root vertex, and h_i is the height of v_i . The sign \pm is +1 if the path from the leaf i to the root goes through the left edge of v_i and -1 if that path goes through the right edge of v_i .

We may now orient P_T by fixing an orientation of the sphere and using the product orientation for $(S^{d-1})^{\times |T|}$. We will call the resulting homology class simply $T \in H_{|T|(d-1)}(\operatorname{Conf}_n(\mathbb{R}^d))$. Note that by its definition, T is in the image of the map from the oriented bordism of $\operatorname{Conf}_n(\mathbb{R}^d)$. In fact, because spheres are stably framed it is in the image of the map from framed bordism, or equivalently stable homotopy.

The relations between these homology classes represent a fundamental blending of geometry and algebra

Proposition 2.7. The classes in $H_*(\operatorname{Conf}(\mathbb{R}^d))$ given by trees satisfy the following relations:

where R, T₁, T₂, and T₃ stand for arbitrary (possibly trivial) subtrees which are not modified in these operations, and $|T_i|$ denotes the number of internal vertices of T_i .

We call this relation the Jacobi identity because of the standard translation between S-trees and bracket expressions, under which this becomes $[[T_1, T_2], T_3] + [[T_2, T_3], T_1] + [[T_3, T_1], T_2] = 0$.

Proof. The anti-symmetry relation follows because the submanifolds defined by these two trees are the same, and their parametrizations differ only by the antipodal map on one factor of S^{d-1} from Definition 2.6 and reordering by moving the factors labeled by vertices of T_1 after those of T_2

The Jacobi identity follows from the existence of Jacobi manifolds who bound the submanifolds in that relation. Letting T be the first tree pictured in the Jacobi identity above, consider the submanifold of $\operatorname{Conf}_n(\mathbb{R}^d)$ defined by conditions (1) and (3) from Definition 2.5, as well as condition (2) for vertices internal to T_1 , T_2 , T_3 or R. For the remaining two vertices we replace condition (2) by

- (2.1) $\sum_{i,j\in\{1,2,3\}} d\left(c(\mathbf{x},T_i),c(\mathbf{x},T_j)\right) = 4\varepsilon^h + 2\varepsilon^{h+1}$, where h is the height of the internal vertex immediately above the subtree R. (2.2) $d\left(c(\mathbf{x},T_i),c(\mathbf{x},T_j)\right) \geq 2\varepsilon^{h+1}$, where $i,j\in\{1,2,3\}$.

Condition (2.1) fixes the perimeter of the triangle with vertices at the centers of the sub-configurations associated to T_1 , T_2 , and T_3 , and condition (2.5) says that triangle must have a minimum side length of at least $2\varepsilon^{h+1}$.

These conditions determine a submanifold J with boundary, whose boundary is where one of the three distances $d\left(c(\mathbf{x},T_i),c(\mathbf{x},T_j)\right)$ is equal to $2\varepsilon^{h+1}$. There are thus three boundary components. By condition (2.1), when some $d\left(c(\mathbf{x},T_i),c(\mathbf{x},T_j)\right)=2\varepsilon^{h+1}$ the remaining center must be distance roughly $2\varepsilon^h$ from the other two. So these components of ∂J are close to being the submanifolds P_T for the T which occur in the Jacobi identity – see Figure 2. We get exactly those manifolds by replacing condition (2.1) by

$$(2.1') d(c(\mathbf{x}, T_i), c(\mathbf{x}, T_j)) = f(\mathbf{x}).$$

Here $f(\mathbf{x})$ is an interpolation function. Its value is $4\varepsilon^h + 2\varepsilon^{h+1}$ when the $c\mathbf{x}, T_i$ form a nearly equilateral triangle. When the configuration in question is in P_T , its value is the total length of the triangle with vertices at $c(\mathbf{x}, T_i)$, namely

$$\sqrt{4\varepsilon^{2h} + \varepsilon^{4h} - 4\varepsilon^{3h}\cos(\theta)} + \sqrt{4\varepsilon^{2h} + \varepsilon^{4h} + 4\varepsilon^{3h}\cos(\theta)} + 2\varepsilon^{h+1},$$

where θ is the angle pictured in Figure 2.



FIGURE 2. The geometry of P_T for T as in the Jacobi identity.

We argue by symmetry that the orientations of the three components of ∂J gives rise to the Jacobi identity exactly. Again referring to Figure 2, for each boundary component we can define an inward normal vector through having $c(\mathbf{x}, T_i)$ and $c(\mathbf{x}, T_j)$ move radially outward, away from their center, and thus needing $c(\mathbf{x}, T_k)$ move radially inward so that condition (2.2) is satisfied. This normal vector is invariant under cyclic permutation of T_1 , T_2 and T_3 , as is the definition of orientation for the P_T for T which appear in the Jacobi identity. Thus, these orientations will either all agree or all disagree with a chosen orientation of J, meaning in either case that the Jacobi identity holds.

Finally, we allow for multiple planetary systems, freeing the points which do not move in the definition of P_T , for example the points x_4 and x_5 in Figure 1.

Definition 2.8. • An *n*-forest is a collection of (by abuse) S-trees, with root vertices at the points $(0,0), (1,0), \ldots$ in the upper half-plane, where each integer from 1 to n labels exactly one leaf.

- If $F = \bigcup T_i$ is a forest, let P_F be the submanifold defined by conditions (2) and (3) of Definition 2.5, replacing condition (1) with $c(\mathbf{x}, T_i) = (i, 0, \dots, 0)$.
- Parameterize P_F by a map of the same name from a product over vertices of F (ordered from left to right by the half-planar embedding of F) of spheres, which when restricted to the coordinates labeled by T_i is a translation of P_{T_i} , namely $P_{T_i} + (i, 0, ..., 0)$.
- By abuse let F denote the homology class represented by P_F , again using our fixed orientation of a sphere to orient the torus and thus its image under P_F .

We recover P_T by letting F be a forest which consists of T and a collection of one-leaf trees. We can summarize our results so far as follows.

Definition 2.9. Let $Pois^d(n)$ denote the quotient of the free module spanned by n-forests by antisymmetry and Jacobi identities as in Proposition 2.7 along with the following:

(commutativity) If F_1 and F_2 consist of the same trees, then $F_1 = \sigma^{(d-1)}F_2$, where σ is the sign of the permutation which relates the ordering of the internal vertices of the trees in F_1 with those of F_2 .

Theorem 2.10. Sending a forest F to the image of the fundamental class of $(S^{d-1})^{\times |F|}$ under P_F gives a well-defined homomorphism from P ois $^d(n)$ to H_* (Conf_n(\mathbb{R}^d)).

Our main theorem will be that this map is an isomorphism (of operads).

2.1. Canonical realization after compactification. There are a number of choices we have made in our definition of P_T , in particular the scale ε . It is tempting to let ε go to zero, which is indeed possible with some recent "compactification technology." There is a canonical completion of these configuration spaces due to Fulton-MacPherson [11] and Axelrod-Singer [2], which we denote $\operatorname{Conf}_n[\mathbb{R}^k]$ in [20]. There we give the following elementary definition.

Definition 2.11. • Define $\alpha_{ij} : \operatorname{Conf}_n(\mathbb{R}^d) \to S^{d-1}$ as sending (x_1, \dots, x_n) to $\frac{x_j - x_i}{|x_j - x_i|}$.

- Let $I = [0, \infty]$, the one-point compactification of the nonnegative reals, and for i, j, k distinct numbers between 1 and n let s_{ijk} : Conf_n(\mathbb{R}^d) $\to I = [0, \infty]$ be the map which sends (x_1, \dots, x_n) to $(|x_i x_j|/|x_i x_k|)$.
- Let $\operatorname{Conf}_n[\mathbb{R}^d]$ be the closer of the image of $\operatorname{Conf}_n(\mathbb{R}^d)$ in

$$(\mathbb{R}^d)^{\times n} \times (S^{d-1})^{\times \binom{n}{2}} \times I^{\times \binom{n}{3}},$$

under the map which is the canonical inclusion on the first factor, the product over all a_{ij} on the second factor, and the product over all s_{ijk} on the third factor.

With such an elementary definition, the hard work is to establish basic properties. This completion is functorial for embeddings and yields a manifold with corners with $\operatorname{Conf}_n(\mathbb{R}^k)$ as its interior, and its strata are naturally indexed by trees (which are not necessarily trivalent). The points added by the boundary may be viewed as "degenerate configurations" in which some number of points now coincide in the large scale but have data which resolves them "infinitesimally." See [20] for a more thorough treatment.

When the submanifold P_T is included in $\operatorname{Conf}_n[\mathbb{R}^k]$, we may send ε in its definition to zero. We leave this as an exercise for the moment, since composing the map P_T with the projections α_{ij} will be a key calculation in Theorem 4.2. After this canonical homotopy of P_T sending ε to zero, the resulting submanifold ends up being the stratum labeled by T in the stratification of $\operatorname{Conf}_n[\mathbb{R}^k]$ as a manifold with corners. Thus these strata represent the homology classes we have been constructing. The Jacobi manifolds are also homotopic to strata which labeled by trees with one four-valent vertex, and their boundaries as strata give rise to the Jacobi identity. Indeed, the manifold with boundary $\operatorname{Conf}_3[\mathbb{R}^k]$ is diffeomorphic to the simplest Jacobi manifold. Thus the compactification $\operatorname{Conf}_n[\mathbb{R}^k]$ "wears its homology on its strata."

2.2. **Historical notes.** To our knowledge, the approach to homology through "orbital systems" first appeared implicitly in Fred Cohen's thesis work ([4], in particular Section 12 of part III). This approach coincides with the "twisted products" constructions from Fadell and Husseini's book ([9], Chapter VI), but in their approach trees and forests do not explicitly appear. The role of trees and forests, and the explicit connection with the theory of operads, has been in the air since the "renaissance" of that theory, in particular [12] which also emphasizes the role of compactifications. One of our aims is to make this basic theory which is well-known to experts as accessible as possible.

3. The cohomology ring

In the previous section we constructed homology classes for the space of Euclidean configurations by mapping in fundamental classes of tori. In this section we pull back cohomology from tori. If homology classes in configuration spaces may be viewed as planetary systems, cohomology classes may be view as recording planetary alignments.

Definition 3.1. Recall $\alpha_{ij} \colon \operatorname{Conf}_n(\mathbb{R}^d) \to S^{d-1}$ as sending (x_1, \dots, x_n) to $\frac{x_j - x_i}{|x_j - x_i|}$. Let $\iota \in H^{d-1}(S^{d-1})$ denote the dual to the fundamental class, using our fixed orientation. Let a_{ij} denote $\alpha_{ij}^*(\iota)$.

The ring generated by these a_{ij} can be represented graphically.

Definition 3.2. Consider graphs with vertices labeled $1, \ldots, n$, with edges which are oriented and ordered. Let $\Gamma(n)$ denote the free module generated by such graphs, which is a ring by taking the union of edges of two graphs in order to multiply them (using the order of multiplication to define the ordering on the union of edges). Map $\Gamma(n)$ to H^* (Conf_n(\mathbb{R}^d)) by sending a generator i^{j} to a_{ij} .

So for example the graph ${}_{4}\mathcal{I}^{2}{}_{1}\mathcal{I}^{3}$, with ${}_{4}\mathcal{I}^{2}$ first in the ordering of edges, is mapped to the product $a_{42}a_{13}$. We will see that the map from $\Gamma(n)$ to $H^{*}\left(\operatorname{Conf}_{n}(\mathbb{R}^{d})\right)$ is surjective. As a base case, we show that after quotienting by the the relation ${}_{i}\mathcal{I}^{j}=(-1)^{d}{}_{i}\mathcal{I}^{i}$, this map is an isomorphism in degree d-1.

Definition 3.3. Let $p_i : \operatorname{Conf}_n(\mathbb{R}^d) \to \operatorname{Conf}_{n-1}(\mathbb{R}^d)$ be the projection map which sends (x_1, \ldots, x_n) to $(x_1, \ldots, \hat{x_i}, \ldots, x_n)$.

Lemma 3.4. The projection map p_i gives $\operatorname{Conf}_n(\mathbb{R}^d)$ the structure of a fiber bundle over $\operatorname{Conf}_{n-1}(\mathbb{R}^d)$, with fiber given by \mathbb{R}^d with (n-1) points removed.

Proof. For simplicity let i=n. Consider a neighborhood $U_{\mathbf{x}}$ of $\mathbf{x}=(x_1,\ldots,x_{n-1})$ of points (y_1,\ldots,y_n) where $d(y_j,x_j)<\epsilon$ for some fixed ϵ less than the minimum of the $\frac{1}{5}d(x_j,x_k)$. Construct a continuous family of homeomorphsims $h_{\mathbf{y}}$ over $y\in U_{\mathbf{x}}$ between $\mathbb{R}^d-\{y_1,\ldots,y_{n-1}\}$ and $\mathbb{R}^d-\{B(x_1,\epsilon),\ldots,B(x_{n-1},\epsilon)\}$, which for good measure is the identity on $\mathbb{R}^d-\{B(x_1,2\epsilon),\ldots,B(x_{n-1},2\epsilon)\}$. This may be done, for example, by "straight-line retractions." A trivialization of this fiber bundle, in other words a homeomorphism between $p_n^{-1}(U_{\mathbf{x}})$ and $U_{\mathbf{x}}\times\mathbb{R}^d-\{x_1,\ldots,x_{n-1}\}$ respecting p_n , is given by

$$(y_1, \cdots, y_n) \mapsto (y_1, \cdots, y_{n-1}) \times h_{\mathbf{x}}^{-1} \circ h_{\mathbf{y}}(y_n).$$

The space \mathbb{R}^d with (n-1) points removed retracts onto $\bigvee_{n-1} S^{d-1}$. We assemble these projection maps into a tower of fibrations first studied by Fadell and Neuwirth [8], which is central in the study of the topology of configuration spaces.

$$\begin{array}{cccc}
& \bigvee_{n-1} S^{d-1} & \longrightarrow \operatorname{Conf}_n(\mathbb{R}^d) \\
& \bigvee_{p_n} \\
& \bigvee_{n-2} S^{d-1} & \longrightarrow \operatorname{Conf}_{n-1}(\mathbb{R}^d) \\
& & \downarrow^{p_{n-1}} \\
& & \vdots \\
& & \bigvee_{2} S^{d-1} & \longrightarrow \operatorname{Conf}_3(\mathbb{R}^d) \\
& & & \bigvee_{p_n} \\
& & & \operatorname{Conf}_2(\mathbb{R}^d) \simeq S^{d-1}
\end{array}$$

These fibrations split. Choice of sections of p_i include adding a new *i*th point "at infinity" or somehow "doubling" the *i*th point.

Proposition 3.5. The first non-trivial homology group $H_{d-1}(\operatorname{Conf}_n)$ is free of rank $\binom{n}{2}$.

Proof. We give a proof only for n > 2. We use the long exact sequences in homotopy of the fibrations constituting the tower of Equation (1), which splits into short exact sequences because the maps p_i admit splittings. We base all of our configuration spaces at the configuration with $x_i = (i, 0, ..., 0)$.

From these short exact sequences, we deduce inductively that $\operatorname{Conf}_i(\mathbb{R}^d)$ is (d-2)-connected, and that $\pi_{d-1}\left(\operatorname{Conf}_i(\mathbb{R}^d)\right)$ is free. Moreover, the rank of $\pi_{d-1}\left(\operatorname{Conf}_i(\mathbb{R}^d)\right)$ is that of $\pi_{d-1}\left(\operatorname{Conf}_{i-1}(\mathbb{R}^d)\right)$ plus i-1, or ultimately $1+2+\cdots+(i-1)$, which is $\binom{i}{2}$. Finally, the Hurewicz theorem applies to give that the homology is isomorphic to homotopy in this dimension.

We now show that the homology and cohomology classes that we have constructed so far generate this first non-trivial group. Let \langle , \rangle denote the standard pairing of cohomology and homology.

Lemma 3.6. Let i < j and $k < \ell$. The pairing $\langle i \mathcal{I}^j, \overset{k}{\dot{\gamma}} \rangle$ is equal to one if $i = k, j = \ell$ and is zero otherwise.

Proof. To evaluate a cycle on i^{j} , it suffices by naturality to map that cycle to S^{d-1} and evaluate it on ι . Because V is the natural image under $P_{k\ell}$ of the fundamental class of S^{d-1} , it suffices to compute the degree of the composite $\alpha_{ij} \circ P_{k\ell} : S^{d-1} \to S^{d-1}$. If i = k and $j = \ell$, then this composite is the identity, which is degree one. If we count the preimages of the "north pole" in S^{d-1} to compute the degree, then we are counting the number of configurations in $P_{k\ell}$ for which x_j is "above" x_i . If either $i \neq k$ or $j \neq \ell$, then at no point will x_j be above x_i , since at least one of them will be stationary, "at infinity", in every configuration parameterized by $P_{k\ell}$.

More generally, to evaluate a cohomology class of $\operatorname{Conf}_n(\mathbb{R}^d)$ represented by some graph, it suffices to count (with signs) the number of points in a cycle for which, for each edge i^{-j} in the graph the point x_j is "above" x_i . Since our cycles P_T are planetary systems, the values of these cohomology classes on them are counting planetary alignments.

Because $H_{d-1}\left(\operatorname{Conf}_n(\mathbb{R}^d)\right)$ is free of rank $\binom{n}{2}$, and this pairing shows that the classes we have defined are linearly independent with the same rank, we have the following.

Corollary 3.7. A basis for $H^{d-1}(\operatorname{Conf}_n(\mathbb{R}^d))$ is given by the classes a_{ij} . The coefficients of a class expressed in this basis are given by evaluating on the homology basis $\binom{k\ell}{l}$ with $k < \ell$.

In general the map from $\Gamma(n)$ to $H^*(\operatorname{Conf}_n(\mathbb{R}^d))$ has relations. If G_1 and G_2 differ by the reversal of k arrows and the reordering of edges as governed by a permutation σ , then

(arrow reversing)
$$G_1 - (-1)^{k(d-1)} (\operatorname{sign} \sigma)^d G_2 = 0$$

Also, because $\iota^2 = 0$ in the cohomlogy of the sphere, any graph with more than one edge between two vertices will map to zero. There is a more subtle relation, which is in some sense dual to the Jacobi identity.

Theorem 3.8. The following relation holds in the image of $\Gamma(n)$ in $H^*(\operatorname{Conf}_n(\mathbb{R}^d))$:

where j, k, and ℓ stand for vertices in the graph which could possibly have other connections to other parts of the graph (indicated by the ends of edges abutting j, k, and ℓ) which are not modified in these operations.

Proof. Using the ring structure, it suffices to consider when there there are no edges incident on j, k and ℓ , other than the two edges involved in the identity. In this case the cohomology classes are all pulled back from $H^{2(d-1)}\left(\operatorname{Conf}_3(\mathbb{R}^d)\right)$ via a map which forgets all x_i except x_j , x_k and x_ℓ . So we may assume that n=3 and $\{j,k,\ell\}=\{1,2,3\}$.

Our proof uses elementary intersection theory to compute some cup products. Since $\operatorname{Conf}_3(\mathbb{R}^k)$ is a manifold, its cohomology is Lefshetz dual to its locally finite homology. Consider the submanifold of $(x_1, x_2, x_3) \in \operatorname{Conf}_3(\mathbb{R}^k)$ such that x_1, x_2 , and x_3 are collinear. This submanifold has three components. Let col_i denote the component in which x_i is in the middle. Since col_i is a properly embedded submanifold of codimension d-1, once oriented it represents a locally finite homology class, which through Lefshetz duality gives rise to a class in $H^{d-1}(\operatorname{Conf}_3(\mathbb{R}^d))$. By Corollary 3.7, this class is determined by its value on the classes Y, which in the context of Lefshetz duality means intersecting col_i with various P_T .

These intersections can be understood directly. The submanifold col_i can only intersect $P_{i,j}$ or $P_{i,k}$, since otherwise x_i would be "at infinity", and thus could not be in the middle of a collinearity. Moreover, $P_{i,j}$ and $P_{i,k}$ each intersect col_i exactly once, namely when x_i is a negative multiple of x_j (respectively, x_k). For purposes of our computation we only need that these intersections differ in sign by -1, coming from orientation reversing of the line on which the three points lie. We deduce that the cohomology class represented by col_i is $\pm (a_{ij} - a_{ik})$.

Under Lefshetz duality, cup products are computed by (transversal) intersections. Since col₁ and col₂ are disjoint, the cohomology classes which they represent cup to zero. We have

$$0 = \pm (a_{12} - a_{13})(a_{23} - a_{21})$$

$$= a_{12}a_{23} - a_{12}a_{21} - a_{13}a_{23} + a_{13}a_{21}$$

$$= a_{12}a_{23} + 0 - (-1)^{d + (d+1)}a_{23}a_{31} + (-1)^{2d}a_{31}a_{12}$$

$$= a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12}.$$

When we translate back to the graphical language, this is exactly the Arnold identity.

This new proof through the disjointness of collinearity submanifolds is using a fundamental geometric observation as the basis for a cohomology ring computation, akin to seeing the cohomology ring of projective spaces through the intersections of linear subspaces. Using cochains defined through collinearities works better for this calculation than our original cochains representing the classes a_{ij} , for which we have that the quadratic polynomial in the Arnold identity is exact but not identically zero (unless d = 2, where Kontsevich observed the vanishing using the differential forms dlog $(x_i - x_j)$.) The collinearity cochains are also invariant under the action of the full group of affine transformations in \mathbb{R}^d .

We will see that there are no further relations among these graph classes in $H^*(\operatorname{Conf}_n(\mathbb{R}^d))$, so that the image of $\Gamma(n)$ will be precisely the following module.

Definition 3.9. Let $Siop^d(n)$ denote the quotient of $\Gamma(n)$ by the arrow-reversing relation and the Arnold identity.

Instead of starting with $\Gamma(n)$, we can restrict to acyclic graphs.

Proposition 3.10. Any element of $Siop^d(n)$ represented by a graph which has a cycle is zero.

Proof. We may use the Arnold identity inductively to reduce to graphs with shorter cycles. But graphs with cycles of length two, that is which have more than one edge between two vertices, are zero. \Box

3.1. **Historical notes.** Analyzing projection maps and in particular assembling them into a tower has been a central tool in this area since its introduction by Fadell and Neuwirth [8]. The calculation of cohomology of configurations in the plane is, famously, due to Arnold [1]. It was generalized in higher dimensions by Cohen [6], to the complements of other collections of subspaces defined by linear equations

by Orlik and Solomon [19], and to a myriad of other contexts. Note that what we call the Arnold identity, along with the fact that generators square to zero, are called the cohomological Yang-Baxter relations in Chapter V of [9], which gives a complete account of the spectral sequence approach to calculation. The graphical notation for this cohomology has been useful for a wide range of recent work, for example [13].

4. The homology-cohomology pairing

Building on the combinatorics which arose in the last two sections, we develop a pairing between graphs and trees which coincides with the evaluation of cohomology on homology of $\operatorname{Conf}_n(\mathbb{R}^d)$.

Definition 4.1. Given an n-graph G and an n-tree T, define the map

$$\beta_{G,T}: \{ \text{edges of } G \} \longrightarrow \{ \text{internal vertices of } T \}$$

by sending an edge i^{j} in G to the vertex at the nadir of the shortest path in T between the leafs with labels i and j. Define the mod-2 configuration pairing of n-graphs G and n-trees T by

$$\left\langle G,\,T\right\rangle = \begin{cases} 1 & \text{if β is a bijection}\\ 0 & \text{otherwise.} \end{cases}$$

Define the dimension-d configuration pairing by, in the first case above, introducing the sign of the permutation relating the orderings of the edges of G and internal vertices of T given in Definition 2.3 when d is even, or $(-1)^k$ where k is the number of edges i^{j} of G for which leaf i is to the right of leaf j under the planar embedding of T when d is odd.

This definition extends to give a pairing between (possibly disconnected) n-graphs G and n-forests F, which is zero if an edge of G has endpoints which label leaves in two different components of F (so that β is not defined).

FIGURE 3. The map $\beta_{G,T}$ for two different trees T. In the first case the configuration pairing is -1 if d is odd or 1 if d is even, and in the second case it is zero.

Theorem 4.2. The homology-cohomology pairing for $\operatorname{Conf}_n(\mathbb{R}^d)$ agrees with the configuration pairing. That is, if we let $\langle -, - \rangle_c$ denote the combinatorially-defined configuration pairing, and let $\langle -, - \rangle_H$ denote the homology-cohomology pairing for $\operatorname{Conf}_n(\mathbb{R}^d)$, then $\langle G, F \rangle_c = \langle G, F \rangle_H$.

Here we have continued the abuse of letting G and F denote both graphs and trees and their images in cohomology and homology of $\operatorname{Conf}_n(\mathbb{R}^d)$.

Proof. For the homology pairing, we must evaluate a product of classes a_{ij} on a submanifold P_F , which by naturality of cap products is equal to computing the degree of the composite

$$\pi_G \circ P_F : \prod_{v \in F} S^{d-1} \xrightarrow{P_F} \operatorname{Conf}_n(\mathbb{R}^d) \xrightarrow{\pi_G} \prod_{e \in G} S^{d-1}.$$

Here π_G is the product over edges of G which associates to $e = i^{j}$ a factor of π_{ij} . By Definitions 2.6 and 3.1, this composite sends $(u_{v_1}, \dots, u_{v_{|F|}})$ to $(\theta_{e_1}, \dots, \theta_{e_{|G|}})$, where for $e = i^{j}$, θ_e is the unit vector

in the direction of

$$\left((n, 0, \cdots, 0) + \sum_{v_k < \text{leaf } i} \pm \varepsilon^{h_k} u_{v_k} \right) - \left((m, 0, \dots, 0) + \sum_{w_\ell < \text{leaf } j} \pm \varepsilon^{h_\ell} u_{w_\ell} \right).$$

Here v_k is the vertex of height k under leaf i, which is in the nth component of the forest F, and similarly w_ℓ is the vertex of height ℓ under leaf j, which is in the mth component of F. If leaves i and j are in the same component, common terms associated to vertices under both leaf i and leaf j cancel, leaving θ_e as the unit vector in the direction of

$$\varepsilon^{h_n} \left(\pm 2u_{v_n} \pm \varepsilon (u_{v_{n+1}} - u_{w_{n+1}}) + \varepsilon^2 \cdots \right),$$

where v_n is the highest vertex under both leaf i and j (if it exists), which is also the nadir of the path between i and j.

Consider the homotopy of P_F , and thus this composite, in which ε approaches zero. Through this homotopy, θ_e approaches either $(\pm 1, 0, \dots, 0)$ if leaves i and j are in different components, or otherwise $\sigma_e u_{v_n}$. From Definition 2.6 we see that σ_e is 1 if leaf i is to the left of j or -1 if it is to the right. Therefore, if there is some edge $i^{\nearrow j}$ in G with leaves i and j in different components of F, then P_F is homotopic to the map between tori with at least one factor the constant map of $(\pm 1, \dots, 0)$, and thus it is of degree zero. Otherwise, P_F is homotopic to the map which sends $(u_{v_1}, \dots, u_{v_{|F|}}) \in \prod_{|F|} S^{d-1}$ to

$$\left(\sigma_{e_1}u_{\beta_{G,F}(e_1)},\ldots,\sigma_{e_{|G|}}u_{\beta_{G,F}(e_{|G|})}\right),\,$$

whose degree agrees with the definition of the configuration pairing through $\beta_{G.F}$.

Because the homology classes of $\operatorname{Conf}_n(\mathbb{R}^d)$ represented by forests satisfy the anti-symmetry and Jacobi identities of Proposition 2.7, and the cohomology classes represented by graphs satisfy arrow-reversing and the Arnold identities, we have geometrically established the following fact, which is established combinatorially in [21].

Corollary 4.3. The dimension-d configuration pairing passes to a well-defined pairing between $\mathcal{P}ois^d(n)$ and $Siop^d(n)$.

We now outline the purely algebraic argument, given in [21], that this pairing between $Pois^d(n)$ and $Siop^d(n)$ is perfect. We only give hints, leaving some fun for the reader.

Lemma 4.4. The module $\mathcal{P}ois^d(n)$ is spanned by n-forests in which all trees are tall (that is, the distance between the leaf with the minimal label and the root is maximal, and that leaf is leftmost in the planar ordering). The module $Siop^d(n)$ is spanned by n-graphs whose components are long (that is, each component is a linear graph, with one endpoint labeled by the minimal label; edges are ordered consecutively and oriented away from the minimal label).



FIGURE 4. A tall tree and a long graph. Here i_1 and j_1 are minimal among indices in the tree and graph respectively.

Sketch of proof. For the forests, use the Jacobi identity inductively to increase the distance from the minimally labeled leaf to the root. For the graphs, use the Arnold identity to reduce the number of edges incident on a given vertex. \Box

The sets of tall forests and long graphs are in one-to-one correspondence with partitions of \mathbf{n} , where both the subsets and the constituent elements of each subset are ordered.

Lemma 4.5. The degree-d configuration pairing of a tall forest and a long graph is equal to one if their associated ordered partitions agree, and is zero otherwise.

Sketch of proof. By definition, the underlying unordered partitions must agree in order for the pairing to be non-zero. When looking at the configuration pairing between a single tall tree T and long graph G which share labels, look at the first place where their orderings differ to see how $\beta_{G,T}$ fails to be a bijection. \square

Thus, on tall forests and long graphs, the configuration pairing is essentially a Kronecker pairing, showing that these spanning sets are linearly independent.

Corollary 4.6. Tall forests form a basis of $Pois^d(n)$. Long graphs form a basis of $Siop^d(n)$. Both $Pois^d(n)$ and $Siop^d(n)$ are torsion-free.

Because tall forests and long graphs form bases, and the dimension-d configuration pairing is a Kronecker pairing on them, we deduce the main algebraic result.

Theorem 4.7. The dimension-d configuration pairing between $Pois^d(n)$ and $Siop^d(n)$ is perfect.

And because the configuration pairing agrees with the homology pairing for $\operatorname{Conf}_n(\mathbb{R}^d)$ on the classes constructed by graphs and forests, we have the following.

Corollary 4.8. The homomorphisms from $\mathcal{P}ois^d(n)$ to $H_*\left(\operatorname{Conf}_n(\mathbb{R}^d)\right)$ and $\operatorname{Siop}^d(n)$ to $H_*\left(\operatorname{Conf}_n(\mathbb{R}^d)\right)$ are injective.

We can now establish the first part of the main result of this paper.

Theorem 4.9. The maps from $\mathcal{P}ois^d(n)$ to $H_*(\operatorname{Conf}_n(\mathbb{R}^d))$ and $Siop^d(n)$ to $H*(\operatorname{Conf}_n(\mathbb{R}^d))$ are isomorphisms.

Proof. We make light use of the Leray-Serre spectral sequence. If in a fibration $F \to E \to B$, we have that B is simply connected and either F or B has torsion-free homology, then this spectral sequence says that $H^*(F) \otimes H^*(B)$ serves as an "upper bound" for $H^*(E)$. That is, if we let $P(X)(t) = \sum (\operatorname{rank} H_i(X))t^i$, the Poincaré polynomial of X, then $P(E) \leq P(F) \cdot P(B)$, where by \leq we mean that this inequality holds for all coefficients of t^i . Moreover, if the homology of F and B are free, equality is achieved only when that of E is free.

Recall the Fadell-Neuwirth tower of fibrations from Equation (1). If d>2 then the long exact sequence for homotopy groups for the fibration $\bigvee_{i=1} S^{d-1} \to \operatorname{Conf}_i(\mathbb{R}^d) \to \operatorname{Conf}_{i-1}(\mathbb{R}^d)$ can be used to inductively establish that these configuration spaces are simply connected. The Leray-Serre spectral sequence upper bound then yields the inequality $P(\operatorname{Conf}_i(\mathbb{R}^d)) \leq P(\operatorname{Conf}_{i-1}(\mathbb{R}^d) \cdot (1+(i-1)t^{d-1})$. For d=2 these spaces are not simply connected - in fact their fundamental groups are pure braid groups almost by definition, and in fact they are classifying spaces for pure braid groups [10] - but Fred Cohen does the extra work at the beginning of Part III of [6] necessary to show that this upper bound still holds.

Inductively we have

$$P\left(\operatorname{Conf}_n(\mathbb{R}^d)\right) \le \prod_{i=1}^{n-1} (1+it^{d-1}).$$

We claim that this upper bound is sharp. Let Q_n be polynomial defined as $Q_n(t) = q_i t^{i(d-1)}$, where q_i is the rank of the submodule of $Siop^d(n)$ with i edges. By Corollary 4.8, Q_n is a lower bound for $P(\operatorname{Conf}_n(\mathbb{R}^d))$, which we compute inductively. The set of long graphs in $Siop^d(i)$ maps to those of $Siop^d(i-1)$ by taking a long graph, removing the vertex labeled n and any edges connected to it, and then reconnecting the two adjacent vertices with a new edge if necessary. Given a long graph G in $Siop^d(i-1)$ there is exactly one

long graph in $Siop^d(i)$ with the same number of edges which maps to it, namely the one in which vertex n is added but not connected to an edge. Moreover, there are i-1 long graphs in $Siop^d(i)$ with one more edge which map to it, since one can choose which of the i-1 vertices in G would have an edge connect to (as opposed to from) the ith vertex. We deduce that $Q_i = Q_{i-1} \cdot (1 + (i-1)t^{d-1})$.

Thus, the lower bound for $H^*(\operatorname{Conf}_n(\mathbb{R}^d))$ given by the submodule $\operatorname{Siop}^d(n)$ matches the upper bound given by the Leray-Serre spectral sequences for the tower of fibrations of Equation (1). We may inductively deduce that $H^*(\operatorname{Conf}_n(\mathbb{R}^d))$ is free, isomorphic to $\operatorname{Siop}^d(n)$. By the Universal Coefficient Theorem and Theorems 4.2 and 4.7, we have that $H_*(\operatorname{Conf}_n(\mathbb{R}^d))$ is isomorphic to $\operatorname{Pois}^d(n)$.

One can also obtain upper bounds by calculations in the cellular homology of the one-point compactifications of these spaces, which is then dual to their chohomology, using a cell structure first developed for d=2 by Fox and Neuwirth [10].

Using the Leray-Serre spectral sequence in full force can lead to some of these results more quickly. For formal reasons, the spectral sequence for each Fadell-Neuwirth fibration collapses, showing immediately that the upper bound on cohomology groups given by each spectral sequence is sharp. One can also use a symmetry argument to deduce the Arnold identity and thus determine the cohomology ring structure. Indeed, these fiber sequences are nice first examples to work with, since even though the group structure mimics that of a trivial (product) fiber sequence, the cohomology ring of $\operatorname{Conf}_n(\mathbb{R}^d)$ differs greatly from that of $\prod_{i=1}^{n-1} (\bigvee_i S^{d-1})$.

In our approach, we not only have an understanding of the homology groups of $\operatorname{Conf}_n(\mathbb{R}^d)$ and the cohomology ring up to isomorphism, but we also have canonical spanning sets and an explicit understanding of the pairing between them, which enables hands-on calculations.

4.1. **Historical notes.** The pairing between graphs and trees we develop (for d odd) was first noticed by Melançon and Reutenauer in a combinatorial study of free Lie superalgebras [17]. It was independently identified as the pairing between canonical spanning sets for homology and cohomology of configuration spaces by Paolo Salvatore, Victor Tourtchine [24], and the author, all within the context of studying spaces of knots. The present paper gives the first full account of this connection, to our knowledge. The pairing is useful in related areas of algebra and topology, such as the study of Hopf invariants [23].

5. Operads

An operad encodes multiplication. Roughly speaking, an operad contains information needed to multiply in an algebra over that operad. For example, in multiplying matrices one must supply an ordering of the matrices to be multiplied, while in multiplying real numbers no such ordering is needed. To Lie multiply (that is, take commutators of) some matrices, one must not only order but parenthesize them.

The many definitions of an abstract operad are necessarily complicated. Even the elegant "an operad is a monoid in the category of symmetric sequences," requires knowing what a symmetric sequence is and then doing some work to relate that definition to standard examples. Thorough introductions to the theory of operads are given elsewhere in this volume. We prefer to be self-contained and to work with operads through trees, so we give our own development here. We start with examples before giving the definition. For now we work with the intuitive definition that an operad \mathcal{O} in a symmetric monoidal category \mathcal{C} is a sequence of objects indexed by natural numbers so that the nth object $\mathcal{O}(n)$ parameterizes ways in which n elements of some kind of algebra (that is, an algebra over that operad) can be multiplied.

- **Examples 5.1.** (1) In any unital symmetric monoidal category \mathcal{C} , the commutative operad $\mathcal{C}om$ has $\mathcal{C}om(n) = \mathbb{1}_{\mathcal{C}}$, since there is only one way to multiply n things commutatively. (For vector spaces, $\mathbb{1}_{\mathcal{C}}$ is the ground field \mathbb{k} ; for spaces, $\mathbb{1}_{\mathcal{C}}$ is a point.)
 - (2) In spaces, the associative operad $\mathcal{A}ss$ has $\mathcal{A}ss(n) = \Sigma_n$, the finite set of orderings of n points, since the product of n things is determined by their order if multiplication is associative. In vector spaces, $\mathcal{A}ss(n) = \mathbb{k}[\Sigma_n]$.

- (3) In vector spaces, the Lie operad $\mathcal{L}ie$ has $\mathcal{L}ie(n)$ spanned by n-trees modulo the anti-symmetry and Jacobi identities of Proposition 2.7 (with d even). In a Lie algebra one must parenthesize elements to multiply them, and our n-trees encode parenthesizations. The anti-symmetry and Jacobi identities are always respected in a Lie algebra, so they appear in the definition of this operad.
- (4) In spaces, the little d-disks operad has $\mathcal{D}isks^d(n)$ the space of n little disks in D^d . Explicitly, $\mathcal{D}isks^d(n)$ is the subspace of $\mathrm{Conf}_n(\mathrm{Int}D^d)\times (0,1]^n$ of $(x_i)\times (r_i)$ such that the balls $B(x_i,r_i)$ are contained in the interior of D^d and have disjoint interiors. This space parameterizes some ways in which maps from S^d , or rather D^d with its boundary going to the basepoint, can be multiplied. Given n maps $f_i:(D^d,\partial)\to (X,*)$ and an element of this space we may multiply the f_i by applying them on the corresponding little disk, sending points outside any little disk to the basepoint of X. See Figure 7.
- (5) We've already seen the degree-d Poisson operad in vector spaces, which has nth entry $\mathcal{P}ois^d(n)$. In a Poisson algebra one can both Lie multiply, represented by trees, and then multiply those results together, represented by placing those trees together in a forest. One can of course also multiply first and then Lie multiply. By the Leibniz rule, the bracket is a derivation with respect to the multiplication, so we may always reduce to expressions which correspond to forests. For example (using bracket notation rather than forests and assuming d is even),

$$\begin{split} [x_1,[x_2,x_3\cdot x_4]] &= [x_1,[x_2,x_3]\cdot x_4]] + [x_1,x_3\cdot [x_2,x_4]] \\ &= [x_1,[x_2,x_3]]\cdot x_4 + [x_1,x_4]\cdot [x_2,x_3] + [x_1,x_3]\cdot [x_2,x_4] + [x_1,[x_2,x_4]]\cdot x_3. \end{split}$$

(6) For any X in C, the endomorphism operad of X has $\mathcal{E}nd_X(n) = \operatorname{Hom}_{\mathcal{C}}(X^{\odot n}, X)$. The endomorphism operad is often too large to understand explicitly, much as groups of homeomorphisms are. But finding an interesting sub-operad of the endomorphism operad will endow X with a multiplication, much as finding a sub-group of the self-equivalences of X gives a group action.

As McClure and Smith point out in [16], the axiomatic definitions of operads follow nicely from reflecting on what they do, just as the axioms of a group all follow from the notion that a group encodes symmetries through a group action. For example, if one has a rule for multiplying four inputs and two inputs, then one can make a rule for multiplying five inputs: first multiply two of them, then take that result as an input with the remaining three original inputs in order to apply the rule for multiplying four inputs. Thus, for any operad there are maps $\mathcal{O}(4) \odot \mathcal{O}(2) \to \mathcal{O}(5)$.

We prefer to state the definition using trees, and, because operads require maps which commute, the language of categories and functors. In this language, what we just said about combining two- and four-input multiplications to make a five-input multiplication is encoded in the first part of Figure 5.

Definition 5.2. • An o-tree is a finite connected acyclic graph with a distinguished vertex called the root. Univalent vertices of an o-tree (not counting the root, if it is univalent) are called leaves.

- At each vertex, the edge which is closer to the root is called the output edge. The edges which are further from the root are called input edges and are labeled from 1 to n.
- At each edge, the vertex of an edge which is further from the root is called its input vertex, and the vertex closer to the root is called its output vertex. We say that one vertex or edge lies over another if the latter is in the path to the root of the former. A non-root edge is called redundant if its initial vertex is bivalent.
- Given an o-tree τ and an edge e, the contraction of τ by e is the tree τ' obtained by identifying the input vertex of e with its output vertex, and removing e from the set of edges. If the label of e was i, the labels of the k edges which were immediately over e will be increased by i-1, and the labels of the edges which shared the output vertex of e with labels greater than i will be increased by k-1.

• Let Υ denote the category whose objects are o-trees, and whose morphisms are generated by contractions of edges (that is, there is a morphism from τ to τ' if τ' is the contraction of τ by e) and relabelings which are isomorphisms (that is, there is a morphism from τ to τ' , which could be τ itself, if τ' is obtained from τ by relabeling of its edges).

See Figure 5 for some examples of objects and morphisms in Υ . Let Υ_n denote the full subcategory of trees with n leaves, which has a terminal object, namely the unique tree with one vertex called the nth corolla γ_n . We allow for the tree γ_0 which has no leaves, only a root vertex, and is the only element of Υ_0 . For a vertex v let |v| denote the number of edges for which v is terminal, usually called the arity of v.

Definition 5.3. An operad is a functor \mathcal{O} from Υ to a symmetric monoidal category (\mathcal{C}, \odot) which satisfies the following axioms.

- (1) $\mathcal{O}(\tau) \cong \odot_{v \in \tau} \mathcal{O}(\gamma_{|v|}).$
- (2) If e is a redundant edge and v is its terminal vertex then $\mathcal{O}(c_{\{e\}})$ is the identity map on $\odot_{v'\neq v}F(\gamma_{v'})$ tensored with the isomorphism $(\mathbb{1}_{\mathcal{C}}\odot -)$ under the decomposition of axiom (1).
- (3) If S is a subtree of τ and if $f_{S,S'}$ and $f_{\tau,\tau'}$ contract the same set of edges, then under the decomposition of (1), $F(f_{\tau,\tau'}) = F(f_{S,S'}) \odot id$.

By axiom (1), the values of \mathcal{O} are determined by its values on the corollas $\mathcal{O}(\gamma_n)$, which corresponds to $\mathcal{O}(n)$ in the usual terminology. Because of the relabeling morphisms, $\mathcal{O}(n)$ has an action by the nth symmetric group. By axiom (3), the values of \mathcal{O} on morphisms may be computed by composing morphisms on sub-trees, so we may identify some subset of basic morphisms through which all morphisms factor. In Figure 5 we illustrate some basic morphisms in Υ . The first corresponds to what are known as \circ_i operations. The second corresponds May's operad structure maps from Definition 1.1 of [14]. That \mathcal{O} is a functor implies the commutativity of diagrams involving these basic morphisms.

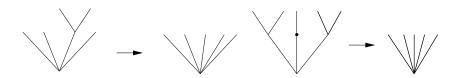


FIGURE 5. Two morphisms in Υ which give rise to standard operad structure maps. The first corresponds to a \circ_i operation, the second to one of May's structure maps.

Filling in what the operad structure maps are for our Examples from 5.1 is a pleasant exercise, which we leave in part to the reader.

Examples 5.4. (1) For Com, all structure maps are the identity.

- (2) For Ass, they are "insertion and relabeling."
- (3) For Lie, the structure maps are defined by grafting trees, that is identifying the root of one with the leaf of another. These are well-defined because the Jacobi and anti-symmetry identities are defined locally.
- (4) For $\mathcal{D}isks^d$ we give a full account. Let T be a tree whose vertices consist of the root vertex v_0 and a terminal vertex v_e for each root edge e. Thus, $T \to \gamma_n$, where n is the number of leaves of T, gives rise to one of May's structure maps as in Figure 5. Given a label $i \in \mathbf{n}$ let v(i) be the initial vertex for the ith leaf, let o(i) be the label of leaf i within the ordering on edges of v(i) and let e(i) be the label of the root edge for which v(i) is terminal. Define $\mathcal{D}isks^d(T \to \gamma_n)$ as follows

$$(x_i^v, r_i^v)_{1 \leq i \leq \#v}^{v \in V(T)} \mapsto (y_j, \rho_j)_{j \in \mathbf{n}} \quad \text{where} \quad y_j = x_{e(j)}^{v_0} + r_{e(j)}^{v_0} x_{o(j)}^{v(j)} \quad \text{and} \quad \rho_j = r_{e(j)}^{v_0} r_{o(j)}^{v(j)}.$$

See Figure 6 for the standard picture.

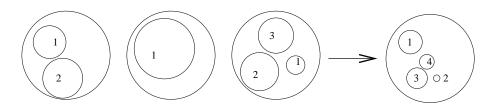


FIGURE 6. A structure map for the 2-disks operad.

(5) In the case of $\mathcal{P}ois^d$, the structure maps are essentially grafting as for $\mathcal{L}ie$, but with an important additional wrinkle given by the Leibniz rule. In order to be precise without unnecessary complication, it helps to switch from forests to more algebraic notation. We may associate to an n-forest an expression in variables x_1, \dots, x_n with two binary products, denoted \cdot and $[\,,\,]$. For example to the

forest $F = \bigvee^{2} \bigvee^{6} \bigvee^{7} \bigvee^{4} \bigvee^{5}$ we associate the bracket expression $[[x_2, x_6], [[x_1, x_7], x_3]] \cdot [x_4, x_5]$. More generally, bracket expressions may include multiplications by \cdot within brackets, but these may be reduced to expressions associated to forests after the Leibniz rule, $[X, Y \cdot Z] = (-1)^{|X||Y|} Y \cdot [X, Z] + [X, Y] \cdot Z$, is imposed.

Let $f: \tau \to \gamma_n$ be a morphism in Υ in which τ is a tree with one internal vertex over the *i*th root edge. Define an operad structure on $\mathcal{P}ois^d = \oplus \mathcal{P}ois^d(n)$ by sending f to the map

$$\mathcal{P}ois^d(n) \otimes \mathcal{P}ois^d(m) \to \mathcal{P}ois^d(n+m-1)$$

where $B_1 \otimes B_2$ is sent to the bracket expression defined as follows. The variables in B_2 are relabeled from x_i to x_m . The variables x_j in B_1 with j > i are re-labeled by x_{j+m-1} . Finally B_2 is substituted for x_i in B_1 . Note that in order to express this in terms of the n-forest basis, the Leibniz rule would then need to be applied repeatedly.

(6) For $\mathcal{E}nd_X$, structure maps are defined by composition.

Finally, we give an anti-climactic definition of an algebra over an operad.

Definition 5.5. An algebra structure for X over an operad \mathcal{O} is a natural transformation of operads $\mathcal{O} \to \mathcal{E} nd_X$.

By adjointness, the maps $\mathcal{O}(n) \to \operatorname{Hom}_{\mathcal{C}}(X^{\odot n}, X)$ give rise to multiplication maps $\mathcal{O}(n) \odot X^{\odot n} \to X$. Because of the relabeling morphisms in Υ , these maps are equivariant with respect to the diagonal symmetric group action on $\mathcal{O}(n) \odot X^{\odot n}$.

As for examples, algebras over Com are commutative algebras, and similar eponymous results hold for Ass, Lie, and $Pois^d$. We will discuss the little disks operad in the next section. As for End_X , the only general statement is that X is an algebra over it.

We leave historical remarks about the theory of operads for other papers in this volume.

6. The homology of the little disks operad

The little disks operad and its action on iterated loop spaces can trace their lineage to the proof of one of the first theorems in homotopy theory, namely that $\pi_2(X)$ is an abelian group. Stated in our language, that proof gives a path of possible multiplications of f and g, two elements of the second loop space $\Omega^2(X)$, starting at $f \cdot g$ and ending at $g \cdot f$. That path lies within the "little rectangles" sub-operad of the space

of all multiplications. If we start with an arbitrary number of maps in any dimension, we are led to the little disks action on a d-fold loop space.

Definition 6.1. The action of $\mathcal{D}isks^d$ on $\Omega^d(X) = \operatorname{Map}\left((D^d, S^{d-1}), (X, *)\right)$ is defined through maps $\mathcal{D}isks^d(n) \times (\Omega^d(X)^{\times n}) \to \Omega^d X$,

which send $\{B_i\} \times \{f_i\}$ to the map whose restriction to B_i is f_i composed with the canonical linear homeomorphism of B_i with D^d . At points in D^d outside of any B_i , the resulting map is constant at the basepoint.

Thus a d-fold loop space is an algebra over the little disks operad. Boardman-Vogt [3] and May [14] showed that the converse is essentially true. That is if X has an action of the little d-disks and $\pi_0(X)$, which is necessarily then a monoid, is in fact a group then X is homotopy equivalent to a d-fold loop space. This result is known as a "recognition principle," since it gives a criterion for recognizing iterated loop spaces.

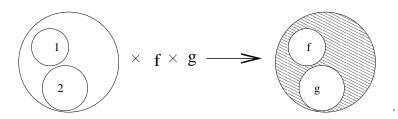


Figure 7. Little 2-disks acting on two maps.

In general, operad actions on spaces are reflected in their homology.

Proposition 6.2. The homology of any operad \mathcal{O} of spaces will be an operad of modules. Moreover, the homology of an algebra over \mathcal{O} will be an algebra over $H_*(\mathcal{O})$.

Proof. That $H_*(\mathcal{O})$ is an operad is immediate by composing the Künneth map

$$H_*(\mathcal{O}(r)) \otimes H_*(\mathcal{O}(n_1)) \otimes \cdots \otimes H_*(\mathcal{O}(n_r)) \to H_*(\mathcal{O}(r) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_r))$$

with the induced map in homology of an operad structure map to get a corresponding operad structure map in homology. Moreover, a map $\mathcal{O} \to \mathcal{E} nd_X$ induces a map $H_*(\mathcal{O}) \to H_*(\mathcal{E} nd_X)$ which in turn maps to $\mathcal{E} nd_{H_*(X)}$, again using the Künneth map.

We can now state the main result of this paper in full.

Theorem 6.3. The homology of the little d-disks operad $\mathcal{D}isks^d$ is the degree d Poisson operad $\mathcal{P}ois^d$. Thus the homology of $\Omega^d(X)$ is an algebra over $\mathcal{P}ois^d$.

Before establishing this theorem, we reflect on its significance, which is to endow the homology of iterated loop spaces a rich additional structure. This homology has intrinsic interest, but may also be used to study homotopy groups. The standard Hurewicz map $\pi_n(X) \to H_n(X)$ is often zero (for example, in all but one degree when X is a sphere). But if we use adjointess to identify $\pi_n(X)$ with $\pi_{n-k}(\Omega^k X)$ then the k-looped Hurewicz map to $H_{n-k}(\Omega^k X)$ can yield additional information. For example, a theorem of Milnor and Moore (whose proof at the end of [18] is left as a nice exercise) states that for rational homotopy and homology, the 1-looped Hurewicz map is injective for simply connected spaces.

We now bring in what we know about configuration spaces through the following.

Lemma 6.4. The space $\mathcal{D}isks^d(n)$ is homotopy equivalent to $Conf_n(\mathbb{R}^d)$.

Proof. Because $\operatorname{Int} D^d$ is homeomorphic to \mathbb{R}^d , their associated configuration spaces $\operatorname{Conf}_n(\operatorname{Int} D^d)$ and $\operatorname{Conf}_n(\mathbb{R}^d)$ are homeomorphic. The space of little disks $\operatorname{Disks}^d(n)$ projects onto $\operatorname{Conf}_n(\operatorname{Int} D^d)$ by definition (mapping to the configuration defined by the centers of the disks). This projection defines a fiber bundle whose fibers, given by the set of possible radii, are convex spaces and thus contractible.

Theorem 4.9 now implies Theorem 6.3 at the level of underlying vector spaces, so we may focus on operad structure. To establish the compatibility between the geometric insertion operad structure of the little disks and the algebraic insertion operad structure of the Poisson operad is easier when considering homology classes of $\mathcal{D}isks^d$ represented by trees rather than forests. The key is to choose appropriately consistent lifts of the submanifolds P_F to the spaces $\mathcal{D}isks^d(n)$ (now the planets really look like planets, being represented by disks rather than points). If for example F is a forest and T is a forest with one component, a \circ_i map would send $\widetilde{P}_F \times \widetilde{P}_T$ precisely to $\widetilde{P}_{F'}$, where F' is the grafting of T onto F. So in homology $F \circ_i T$ is the grafting of T onto the ith leaf of F accordingly.

To manage the general case, we focus on cohomology instead of homology. It is geometrically easier to establish the linear dual of Theorem 6.3, identifying the cohomology of $\mathcal{D}isks^d$ with the cooperad structure on $\mathcal{S}iop^d$. Translating to homology is then a matter of pure algebra and combinatorics, which is carried out in [21]. A cooperad is a functor from Υ^{op} to a symmetric monoidal category which satisfies axioms dual to those of Definition 5.3. Note that in associating a dual cooperad to an operad, we are not changing the symmetric monoidal product. For example, a standard cooperad in the category of vector spaces is defined using the tensor product rather than the direct sum.

Definition 6.5. To an o-tree τ with n leaves and two distinct integers $j, k \in \mathbf{n}$ let v be the nadir of the shortest path between leaves labelled i and j and define $J_v(j), J_v(k)$ to be the labels of the branches of v over which leaves j and k lie.

The module $Siop^d$, forms a cooperad which associates to the morphism $\tau \to \gamma_n$ the homomorphism g_τ sending $G \in \Gamma$ to $(\text{sign } \pi)^d \bigotimes_{v_i} G_{v_i}$. Here v_i ranges over internal vertices in τ and $G_{v_i} \in \Gamma^{|v_i|}$, is defined by having for each edge in G, say from j to k, an edge from $J_v(j)$ to $J_v(k)$ in G_v . The edges of G_v are ordered in accordance with that of the edges in G which give rise to them, and π is the permutation relating this order on all of the edges in $\bigotimes_{v_i} G_{v_i}$ to the ordering within G.

Consider for example when τ is the first tree from Figure 5, with leaf labeling given by the planar embedding. The corresponding cooperad structure map would send a single graph G on five vertices to the tensor product of two graphs, G_r with four vertices and G_v with two vertices. The graph G_v would have an edge between its two vertices if and only if G had ${}_{3} \nearrow^{4}$ as an edge. Any edge of G of the form ${}_{1} \nearrow^{3}$ or ${}_{1} \nearrow^{4}$ would give rise to an edge ${}_{1} \nearrow^{3}$ in G_1 . The edge ${}_{5} \nearrow^{4}$ in G would give rise to ${}_{4} \nearrow^{3}$ in G_1 .

Theorem 6.6 (Thm. 6.8 of [21]). The cooperad structure on $Siop^d$ is linearly dual to that of $Pois^d$ through the configuration pairing.

The key to proving this theorem is that the configuration pairing can be defined directly on bracket expressions. Looking at Definition 4.1, we use innermost pairs of brackets instead of nadirs of paths to define the analogue of the map $\beta_{G,T}$. Remarkably, the Leibniz rule is respected by this extended definition of this pairing. Indeed, the configuration pairing can be viewed as the central algebraic object in this area, and the anti-symmetry, Jacobi, Leibniz, and Arnold identities arise naturally in describing its kernel.

While the operad structure of $\mathcal{P}ois^d$ is more familiar, the cooperad structure on $\mathcal{S}iop^d$ is simpler. The operad maps on $\mathcal{P}ois^d(n)$ require the Leibniz rule to be applied recursively to reduce to any standard basis, while the cooperad maps on $\mathcal{S}iop^d$ require no such reduction for many standard bases. While here we use this cooperad as a useful way to prove a theorem about the corresponding operad, in work on Koszul duality cooperads play an equal role.

Proof of Theorem 6.3. By Lemma 6.4 and Theorem 4.9, $H_*(\mathcal{D}isks^d)$ and $\mathcal{P}ois^d$ are isomorphic as vector spaces, so it suffices to consider their operad structure. By Theorem 4.7 and Theorem 6.6, we may instead establish that the cooperad structures on $\mathcal{S}iop^d$ and the cohomology of $\mathcal{D}isks^d$ agree.

Let $f: \tau \to \gamma_n$ be a morphism in Υ , where γ_n is a corolla, and let $\prod_{w \in \tau} \mathcal{D}isks^d(|w|) \to \mathcal{D}isks^d(n)$ be the corresponding operad structure map. Using the ring structure on cohomology, it suffices to understand the pullback of a generator a_{ij} . By definition, we consider the composite

$$\pi_{ij} \circ \mathcal{D}isks^d(f) : \prod_{w \in \tau} \mathcal{D}isks^d(|w|) \to \mathcal{D}isks^d(n) \to S^{d-1}.$$

We apply a homotopy in which at time t the disks in $\mathcal{D}isks^d(|w|)$ are all scaled by by t^h where h is the height of the vertex w. As t approaches zero, $\pi_{ij} \circ \mathcal{D}isks^d(f)$ approaches projection onto the factor of $\mathcal{D}isks^d(|v|)$ where v is the nadir of the shortest path between leaves labelled i and j, followed by $\pi_{J_v(i)J_v(j)}$, as in Definition 6.5. Thus a_{ij} pulls back to $a_{J_v(i)J_v(j)}$ in the vth factor of $\mathcal{D}isks^d$, in agreement with the cooperad structure on $\mathcal{S}iop^d$.

To recap, we have now shown that the homology of a d-fold loop space is a Poisson algebra. The multiplication is the standard one given by loop-sum. The bracket, known as the Browder bracket, reflects some "higher commutativity" of the loop-sum. If two homology classes are represented by (pseudo-)manifolds $M, N \to \Omega^d X$, then their bracket will be represented by the map $S^{d-1} \times M \times N \to \Omega^d X$ which when restricted to $v \times M \times N$ "multiplies M and N in the direction of v."

6.1. **Historical notes.** The operad structure on spaces of little disks was determined by Cohen in his thesis [6]. There both the non-equivariant and the much more delicate equivariant homology of these spaces are determined, though the language of operads is not employed. Equivariant homology classes yield operations in the homology of iterated loop spaces [7], which are algebras over this operad, which is a main focus of [6] and [15]. The simplest example is with coefficients modulo two, where for any homology class x, we have [x, x] = 0. But this class can be "divided by two, using only a hemisphere's worth of Browder multiplication." The result as an operation which "sends x to $\frac{[x, x]}{2}$."

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